

My question to ChatGPT: (on July 30, 2025)

In three-dimensional space a bathtub vortex can have only one shape and run clockwise or counter clockwise. What are all the possible variations of rotation in four-dimensional space?

Its answer:

You're now digging into the **rotation group of 4D space**, which leads directly into the mathematics of **SO(4)** — the group of rotations in 4D Euclidean space.

This opens up a much richer landscape than in 3D. Let's go step-by-step.

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## 1. Rotations in 3D (SO(3)) – The Bathtub Vortex

In 3D, all rotations happen **around a single axis**. That means:

- Any rotation can be described by:
  - **One plane of rotation** (perpendicular to the axis),
  - **One angular velocity**, and
  - **One direction** (clockwise or counterclockwise).
- Result: a vortex like a bathtub drain can spin **clockwise** or **counterclockwise**, but always in a single plane.

So, **1 shape × 2 directions = 2 variants**.

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## 2. Rotations in 4D (SO(4)) – Richer Possibilities

In 4D, the situation is **fundamentally different**.

### ! Key Fact:

In 4D, every rotation can be decomposed into **two simultaneous independent rotations** in **two orthogonal planes**.

This is unique to 4D — it doesn't happen in 3D or 2D.

That means a vortex-like structure in 4D (such as a “4D bathtub vortex”) can spin:

- In **plane A** at angular velocity  $\omega_1$ ,  
 $\omega_1$
- And in **plane B** at angular velocity  $\omega_2$ ,  
 $\omega_2$

- And the planes are **independent** and **orthogonal**.

This is known as a **double rotation**.

### 3. Types of Rotation in 4D

Let's classify the possible rotation patterns:

Type	Description	Degrees of Freedom
<b>Simple rotation</b>	Rotation in a single plane; no motion in the orthogonal plane	Like 3D rotation
<b>Double rotation</b>	Independent rotation in two orthogonal planes	Unique to 4D
<b>Equal double rotation</b>	Both planes rotate at same speed	Special symmetry
<b>Opposite double rotation</b>	One clockwise, one counter-clockwise	Possible in 4D only
<b>Clifford rotation</b>	All directions rotate uniformly — maximal symmetry	Think hypersphere spin
<b>Degenerate rotation</b>	One plane rotates, the other has zero rotation	Reduces to simple

### 4. Visualizing the Planes of Rotation

In 4D, you can choose rotation planes among:

- x-y
- x-z
- x-w
- y-z
- y-w
- z-w

You pick **two orthogonal planes**. That's what gives rotation its multidimensional character.

There are  $\binom{6}{2} = 15$  possible plane pairs, but only **3 mutually orthogonal planes exist** for a 3D object embedded in 4D. So, most structures involve **2 planes at most**.

### 5. Total Variations of 4D Vortex Rotation

Each 4D vortex rotation can be classified by:

1. **Choice of two planes** → finite possibilities (from dimension constraints).

2. **Direction in each plane** → clockwise or counterclockwise (×2).
3. **Speed in each plane** → any real positive value.
4. **Relative angular velocities** → matched, mismatched, opposing.

**Infinite total variations** due to continuous speed values, but **discrete classes** of behavior:

Class	Rotation Pattern
Single-plane (simple)	Like 3D vortex, but only in one 2D slice
Dual same-direction	Both planes spin in the same sense
Dual opposite-direction	One clockwise, one counter-clockwise
Symmetric (equal-speed)	Clifford-like uniform rotation
Asymmetric (unequal-speed)	General double rotation

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## Summary

In 3D:

- One rotation plane,
- One spin direction,
- Two total vortex variants.

In 4D:

- **Two independent rotation planes,**
- Each with:
  - Direction: clockwise or counterclockwise,
  - Speed: continuous range,
- **Rotation types:** simple, double, symmetric, asymmetric, opposing.

 **Result:**

**Infinitely many rotational variations** of a 4D vortex, falling into **several distinct classes**, including types that have **no 3D analog**.

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