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My question to ChatGPT: (on July 30, 2025)

An ordinary smoke ring takes the form of a torus rotating around its ring axis. How many forms can a 4D smoke ring have? Please describe all the variations and leave none out.

Its answer:

Title: Shapes and Structures of 4D Smoke Rings

An ordinary 3D smoke ring takes the form of a torus (topologically $S^2 \times S^1$ or embedded as a torus in \mathbb{Z}^3) rotating around its ring axis. Generalizing this idea to four-dimensional (4D) space offers a rich set of geometric and topological possibilities. However, it is essential to ground this exploration in proper mathematical frameworks and physical interpretation.

1. Standard 4D Smoke Ring (Topological Model: $S^1 \times S^2$)

Description: The most direct analog of a 3D smoke ring in 4D is a 3-manifold with the topology $S^1 \times S^2$, representing a 2-sphere (S^2) swept along a circular path (S^1).

Embedding: This object is a 3D hypersurface embedded in 4D Euclidean space (\mathbb{Z}^4).

Rotation: It may rotate simultaneously in multiple planes (degrees of freedom increase in 4D), such as in the S^1 direction or around the S^2 fibers.

2. Fibered or Twisted Bundles

Description: One may consider non-trivial fiber bundles of S^2 over S^1 . There are only two such bundles: the trivial one ($S^1 \times S^2$) and a non-orientable version.

Geometry: These may introduce different global twisting structures but do not imply exotic topology.

3. Linked Smoke Rings in 4D

Description: In 4D, multiple $S^1 \times S^2$ structures can be embedded such that they are linked, using the extra dimension to allow non-intersecting yet interdependent paths.

Note: Linking phenomena are more permissive in 4D than in 3D.

4. Multiple Rings (Disjoint Copies)

Description: Multiple smoke rings can exist independently in 4D space. These should be treated as disjoint embeddings of $S^1 \times S^2$, not as a product manifold.

5. Higher Genus Structures

Description: One could replace the spherical fiber (S^2) with higher-genus surfaces (Σ_x), forming 3-manifolds like $S^1 \times \Sigma_x$. These represent more topologically complex ring-like structures.

Interpretation: Such configurations are speculative for physical smoke rings but are mathematically plausible embeddings.

6. Curved or Geometrically Varied Embeddings

Description: The embedding of $S^1 \times S^2$ in \mathbb{Z}^4 can involve intrinsic or extrinsic curvature, affecting its appearance and motion.

Note: These are not topologically distinct but may differ geometrically.

7. Knotted 2D Surfaces

Caution: The knotting of 1D loops in 4D space is trivial (all knots can be undone), but the embedding of 2D surfaces like S^2 or $S^1 \times S^2$ can be nontrivial in 4D.

Example: A knotted S^2 surface in \mathbb{Z}^4 could represent an exotic configuration of a smoke ring, though such knots are complex and not yet fully classified.

8. Exotic Smooth Structures

Clarification: Exotic smooth structures occur in dimension 4 (notably \mathbb{R}^4) but do not affect $S^1 \times S^2$, which has a unique smooth structure.

Summary

Valid topological and geometric configurations for 4D analogs of smoke rings include:

The standard product $S^1 \times S^2$

Non-trivial bundles (limited to one known variant)

Linked/disjoint embeddings of $S^1 \times S^2$ in \mathbb{Z}^4

Generalizations using higher-genus surfaces

Curved or geometrically varied embeddings

The above models capture plausible ways a smoke ring-like structure could exist or behave in four-dimensional space. Caution is advised when extending 3D analogies too literally or applying mathematical terms without precision.

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My Question:

Are you saying there are a total of only 8 possible forms of a 4D smoke ring?

Its answer:

No, the document does **not** claim there are *only* 8 possible forms of a 4D smoke ring.

Instead, it outlines **8 categories** of plausible configurations based on known mathematical structures and physical analogies. These categories are **not exhaustive** in the strictest sense but aim to organize the most conceptually and mathematically sound possibilities. Here's what that means:

The 8 Forms Are Representative, Not Exhaustive

Each item (e.g. “Standard 4D Smoke Ring,” “Linked Rings,” etc.) is a **broad class** that may contain **infinitely many specific instances or variations**, such as:

- Different embeddings of $S^1 \times S^2 \times S^1 \times S^2$
- Various geometric deformations
- Complex linking patterns
- Different genus surfaces in $S^1 \times \Sigma_g$

So while there are 8 **types** listed, the total number of unique shapes and behaviors under each type is potentially **uncountable**, especially when considering geometry, curvature, and dynamic behaviors in 4D space.

Why No Complete Enumeration Exists

A complete, finite classification of “all” 4D smoke ring forms is currently impossible because:

- Topology in 4D is extremely rich and only partially classified.
 - The embedding of 3-manifolds into 4D space has deep unresolved questions.
 - Physical realizations of such objects (like with real 4D fluids) are purely hypothetical.
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Summary Answer to Your Question:

Are there exactly 8 possible forms of a 4D smoke ring?

No. There are 8 **categories of known and plausible forms**, but within each category are potentially **infinitely many distinct configurations**.

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My Question:

For comparison, how many forms can an ordinary 3D smoke ring take?

Its answer:

The number of **forms a 3D smoke ring can take** depends on how you define "form" — topologically, geometrically, dynamically, or physically. Here's a structured breakdown:



1. Topological Form

- **Answer: One** — it is topologically a **torus**, $S^1 \times S^1$.
 - Any ordinary smoke ring in 3D, viewed as a continuous loop of vorticity, is homeomorphic to a torus regardless of its exact shape or thickness.
 - **So topologically, there's only one form.**
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2. Geometric Forms (Shape in Space)

- **Answer: Infinitely many.**
 - A 3D smoke ring can vary in:
 - **Major radius vs. minor radius** (thin vs. fat),
 - **Oblateness or elongation**,
 - **Cross-sectional deformation**,
 - Whether it is **circular, elliptical, wavy, or squashed**.
 - These are geometrically different but topologically equivalent.
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3. Dynamical Forms (Motion and Behavior)

- **Answer: Many**, potentially **uncountably many**, depending on flow dynamics.
- Smoke rings may differ in:
 - **Rotation speed** (solid body-like vs. shearing),
 - **Vorticity distribution** (uniform, Gaussian, etc.),

- **Instability** (e.g., breakdown into smaller vortices),
- **Interaction with surroundings** (walls, other rings, turbulence).
- These are **dynamical states**, not static shapes.

4. Knotted or Linked Vortex Rings

- **Answer: Infinitely many.**
- Smoke rings (or more generally vortex rings) **can** be knotted or linked, though these are rare or engineered cases (not typical puffs of smoke).
 - **Knotted vortex ring** = vortex filament shaped like a knot (e.g., trefoil).
 - **Linked rings** = two or more rings intertwined (e.g., Hopf link).
- This opens up **entire families** of knotted and linked smoke/vortex structures.

5. Cross-Sectional Topology

- Smoke rings typically have a circular cross-section, but you could imagine more complex internal structures (e.g., hollow core, multiple vortex filaments) — though such variations are theoretical or experimental.

Summary Table

Interpretation	Number of Forms	Notes
Topological	1	Always a torus $S^1 \times S^1$
Geometric (Shape)	∞	Vary by size, distortion, deformation
Dynamical (Behavior)	∞	Vorticity, motion, breakdowns, interactions
Knotted/Linked Vortices	∞	Vortex core forms can be knotted or linked
Physical Realizations	∞ (but constrained)	By fluid type, boundary conditions, medium, etc.

In Short:

There is one topological form of a 3D smoke ring (a torus), but infinitely many geometric and dynamical variations, including exotic cases like knotted or linked vortex rings.